e content for students of patliputra university

B. Sc. (Honrs) Part 2paper 3

Subject:Mathematics

Title/Heading of topic:Test for convergence of infinite series

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2.3 Tests for the convergence of infinite series

1. Comparision Test:

Let $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ be two positive term series such that

$u_n \leq kv_n \ \forall \ n$ (where k is a positive number)

Then (*i*) If $\sum_{n=1}^{\infty} v_n$ converges then $\sum_{n=1}^{\infty} u_n$ also converges.

(*ii*) If $\sum_{n=1}^{\infty} u_n$ diverges then $\sum_{n=1}^{\infty} v_n$ also diverges.

Example 4 Test the convergence of the following series

$(i)\sum_{n=1}^{\infty} \frac{1}{n^n}$ $\quad (ii)\sum_{n=2}^{\infty} \frac{1}{log\ n}$ $\quad (iii)\sum_{n=1}^{\infty} \frac{1}{2^n+x} \forall x > 0$

Solution: (*i*) Here $u_n = \frac{1}{n^n}$ We know that $n^n > 2^n$ for $n > 2$

Hence $\frac{1}{n^n} < \frac{1}{2^n}$ for $n > 2$

Now $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series $(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots)$ whose common ratio is $\frac{1}{2}$.

Since $\frac{1}{2} < 1 \therefore \sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series. Thus by comparision test $\sum_{n=1}^{\infty} \frac{1}{n^n}$ is also convergent.

(*ii*) Here $u_n = \frac{1}{\log n}$ We know that $\log n < n$ for $n \geq 2$

Hence $\frac{1}{\log n} > \frac{1}{n}$ for $n \geq 2 \quad \Rightarrow \frac{1}{n} < \frac{1}{log\ n}$ for $n \geq 2$

Now $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (As $\mathrm{p} = 1$) . Thus by comparision test $\sum_{n=2}^{\infty} \frac{1}{\log n}$ is also divergent.

(*iii*) Here $u_n = \frac{1}{2^n+x}$. Clearly $2^n + x > 2^n$ $(as\ x > 0)$

$\therefore \frac{1}{2^n+x} < \frac{1}{2^n}$

Now $\sum_{n=1}^{\infty} \frac{1}{2^n}$ is a geometric series $(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots)$whose common ratio is $\frac{1}{2}$.

Since $\frac{1}{2} < 1 \therefore \sum_{n=1}^{\infty} \frac{1}{2^n}$ is a convergent series. Thus by comparision test $\sum_{n=1}^{\infty} \frac{1}{2^n+x}$ is also convergent.

Example 4 Test the convergence of the series$\sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$

Solution: Here $u_n = \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$

Clearly $\frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{1}{n^2} + \frac{1}{n^2} < \frac{1}{n^2}$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series (As p $= 2 > 1$) . Thus by comparision test $\sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$ is also convergent.

2. Limit Form Test:

Let $\sum_{n=1}^{\infty} u_n$and $\sum_{n=1}^{\infty} v_n$be two positive term series such that

$\lim_{n\to\infty} \frac{u_n}{v_n} = l$ (where l is a finite and non zero number).

Then $\sum_{n=1}^{\infty} u_n$and $\sum_{n=1}^{\infty} v_n$behave in the same manner i.e. either both converge or both diverge.

Example 5 Test the convergence of the series $\frac{1}{3.7} + \frac{1}{4.9} + \frac{1}{5.11} + \ldots..$

Solution: Here $u_n = \frac{1}{(n+2)(2n+5)}$

Let $v_n = \frac{1}{n^2}$. Now consider $\frac{u_n}{v_n} = \frac{1}{(n+2)(2n+5)} n^2 = \frac{n^2}{2n^2+9n+10}$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{2n^2+9n+10}$

$= \lim_{n\to\infty} \frac{1}{2+\frac{9}{n}+\frac{10}{n^2}} = \frac{1}{2}$ (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{(n+2)(2n+5)}$ also converges.

Example 6 Test the convergence of the series

$(i) \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \ldots . . (ii) \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

Solution:(*i*) Here $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$

Let $v_n = \frac{1}{\sqrt{n}}$. Now consider $\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$$

$$= \lim_{n\to\infty} \frac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}}$$

$= \frac{1}{\sqrt{2}}$ (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (as $p = \frac{1}{2} < 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$ also diverges.

(*ii*) Here $u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

$$= \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \cdot \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}} = \frac{(n+1)-(n-1)}{n\sqrt{n+1}+\sqrt{n-1}} = \frac{2}{n\sqrt{n+1}+\sqrt{n-1}}$$

Let $v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$. Now consider $\frac{u_n}{v_n} = \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}}$

$$=\lim_{n\to\infty}\frac{1}{2+\frac{9}{n}+\frac{10}{n^2}}=\frac{1}{2}$$ (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{(n+2)(2n+5)}$ also converges.

Example 6 Test the convergence of the series

$(i)\frac{1}{\sqrt{2}+\sqrt{3}}+\frac{1}{\sqrt{3}+\sqrt{4}}+\frac{1}{\sqrt{4}+\sqrt{5}}+\ldots..(ii)\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

Solution:(i) Here $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$

Let $v_n = \frac{1}{\sqrt{n}}$. Now consider $\frac{u_n}{v_n} = \frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$

$$\Rightarrow \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}+\sqrt{n+2}}$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}}$$

$$= \frac{1}{\sqrt{2}}$$ (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges (as $p = \frac{1}{2} < 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$ also diverges.

(ii) Here $u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

$$=\frac{\sqrt{n+1}-\sqrt{n-1}}{n}\cdot\frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}} = \frac{(n+1)-(n-1)}{n\sqrt{n+1}+\sqrt{n-1}} = \frac{2}{n\sqrt{n+1}+\sqrt{n-1}}$$

Let $v_n = \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$. Now consider $\frac{u_n}{v_n} = \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}}$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}}$$

$$= \lim_{n\to\infty} \frac{2}{\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}}$$

$= 1$ (which is a finite and non zero number)

Hence by Limit form test , $\sum_{n=1}^{\infty} u_n$ and $\sum_{n=1}^{\infty} v_n$ behave similarly.

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (as $p = \frac{3}{2} > 1$)

$\therefore \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ also converges.

Example 7 Test the convergence of the series

(*i*) $\sum_{n=1}^{\infty} \left[(n^3+1)^{1/3} - n\right]$ (*ii*) $\sum_{n=1}^{\infty} \sin\frac{1}{n}$

Solution:(*i*) Here $u_n = (n^3+1)^{1/3} - n = n\left(1+\frac{1}{n^3}\right)^{1/3} - n$

$$= n\left[1 + \frac{1}{3n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!} \cdot \frac{1}{n^6} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!} \cdot \frac{1}{n^9} + \cdots\right] - n$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5}$$

Let $v_n = \frac{1}{n^2}$.

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{3}$ (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (as $p = 2 > 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also converges (by Limit form test).

(*ii*) Here $u_n = \sin\frac{1}{n}$. Let $v_n = \frac{1}{n}$.

Then $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$

=1 (which is a finite and non zero number)

Since $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges (as $p = 1$)

$\therefore \sum_{n=1}^{\infty} u_n$ also diverges (by Limit form test).